$$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s), \tag{A.22}$$

which is the desired result.

EXAMPLE A.8 Time Product of Sinusoidal Signal

Find the Laplace transform of $f(t) = t \sin \omega t$.

Solution. The Laplace transform of $\sin \omega t$ is

$$\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}.$$

Hence, using Eq. (A.22), we obtain

$$F(s) = -\frac{d}{ds}\left[\frac{\omega}{s^2+\omega^2}\right] = \frac{2\omega s}{(s^2+\omega^2)^2}.$$

The following commands in MATLAB yields the same result,

```
syms s t w
laplace(t*sin(w*t)).
```

A.1.2 Inverse Laplace Transform by Partial-Fraction Expansion

As we saw in Chapter 3, the easiest way to find f(t) from its Laplace transform F(s), if F(s) is rational, is to expand F(s) as a sum of simpler terms that can be found in the tables via partial-fraction expansion. We have already discussed this method in connection with simple roots in Section 3.1.5. In this section, we discuss partial-fraction expansion for cases of complex and repeated roots.

Complex Poles In the case of quadratic factors in the denominator, the numerator of the quadratic factor is chosen to be first order as shown in Example A.9. Whenever there exists a complex conjugate pair of poles such as

$$F(s) = \frac{C_1}{s-p_1} + \frac{C_2}{s-p_1^*},$$

we can show that

$$C_2 = C_1^*$$

(see Problem 3.1) and that

$$f(t) = C_1 e^{p_1 t} + C_1^* e^{p_1^* t} = 2\mathrm{Re}(C_1 e^{p_1 t}).$$

Assuming that $p_1 = \alpha + j\beta$, we may rewrite f(t) in a more compact form as

$$\begin{aligned} f(t) &= 2\mathrm{Re}\{C_1 e^{p_1 t}\} = 2\mathrm{Re}\{|C_1| e^{j\arg(C_1)} e^{(\alpha+j\beta)t}\} \\ &= 2|C_1| e^{\alpha t}\cos[\beta t + \arg(C_1)]. \end{aligned} \tag{A.23}$$

EXAMPLE A.9 Partial-Fraction Expansion: Distinct Complex Roots

Find the function f(t) for which the Laplace transform is

$$F(s) = \frac{1}{s(s^2+s+1)}.$$

Solution. We rewrite F(s) as

$$F(s) = \frac{C_1}{s} + \frac{C_2 s + C_3}{s^2+s+1}.$$

Using the cover-up method, we find that

$$C_1 = sF(s)|_{s=0} = 1.$$

Setting $C_1 = 1$ and then equating the numerators in the partial-fraction expansion relation, we obtain

$$(s + s + 1) + (C_2 s + C_3)s = 1.$$

After solving for C_2 and C_3, we find that $C_2 = -1$ and $C_3 = -1$. To make it more suitable for using the Laplace transform tables, we rewrite the partial fraction as

$$F(s) = \frac{1}{s} - \frac{s+\frac{1}{2}+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}.$$

From the tables we have,

$$\begin{aligned} f(t) &= \left(1 - e^{-t/2}\cos\sqrt{\frac{3}{4}}\,t - \frac{1}{\sqrt{3}}e^{-t/2}\sin\sqrt{\frac{3}{4}}\,t\right)1(t) \\ &= \left(1 - \frac{2}{\sqrt{3}}e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t - \frac{\pi}{6}\right)\right)1(t). \end{aligned}$$

Alternatively, we may write F(s) as

$$F(s) = \frac{C_1}{s} + \frac{C_2}{s-p_1} + \frac{C_2^*}{s-p_1^*}, \qquad \text{(A.24)}$$

where $p_1 = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$. $C_1 = 1$, as before, and now

$$\begin{aligned} C_2 &= (s-p_1)F(s)|_{s=p_1} = -\frac{1}{2} + j\frac{1}{2\sqrt{3}}, \\ C_2^* &= -\frac{1}{2} - j\frac{1}{2\sqrt{3}}, \end{aligned}$$

and