# Lecture 10
# Solution via Laplace transform and matrix exponential

- Laplace transform
- solving $\dot{x} = Ax$ via Laplace transform
- state transition matrix
- matrix exponential
- qualitative behavior and stability

# Laplace transform of matrix valued function

suppose $z : \mathbf{R}_+ \to \mathbf{R}^{p \times q}$

**Laplace transform:** $Z = \mathcal{L}(z)$, where $Z : D \subseteq \mathbf{C} \to \mathbf{C}^{p \times q}$ is defined by

$$Z(s) = \int_0^\infty e^{-st} z(t)\, dt$$

- integral of matrix is done term-by-term
- convention: upper case denotes Laplace transform
- $D$ is the *domain* or *region of convergence* of $Z$
- $D$ includes at least $\{s \mid \Re s > a\}$, where $a$ satisfies $|z_{ij}(t)| \leq \alpha e^{at}$ for $t \geq 0$, $i = 1, \ldots, p$, $j = 1, \ldots, q$

# Derivative property

$$\mathcal{L}(\dot{z}) = sZ(s) - z(0)$$

to derive, integrate by parts:

$$\begin{aligned}
\mathcal{L}(\dot{z})(s) &= \int_0^\infty e^{-st}\dot{z}(t)\,dt \\
&= e^{-st}z(t)\Big|_{t=0}^{t\to\infty} + s\int_0^\infty e^{-st}z(t)\,dt \\
&= sZ(s) - z(0)
\end{aligned}$$

# Laplace transform solution of $\dot{x} = Ax$

consider continuous-time time-invariant (TI) LDS

$$\dot{x} = Ax$$

for $t \geq 0$, where $x(t) \in \mathbf{R}^n$

- take Laplace transform: $sX(s) - x(0) = AX(s)$
- rewrite as $(sI - A)X(s) = x(0)$
- hence $X(s) = (sI - A)^{-1}x(0)$
- take inverse transform

$$x(t) = \mathcal{L}^{-1}\left((sI - A)^{-1}\right)x(0)$$

# Resolvent and state transition matrix

- $(sI - A)^{-1}$ is called the *resolvent* of $A$

- resolvent defined for $s \in \mathbf{C}$ *except* eigenvalues of $A$, $i.e.$, $s$ such that $\det(sI - A) = 0$

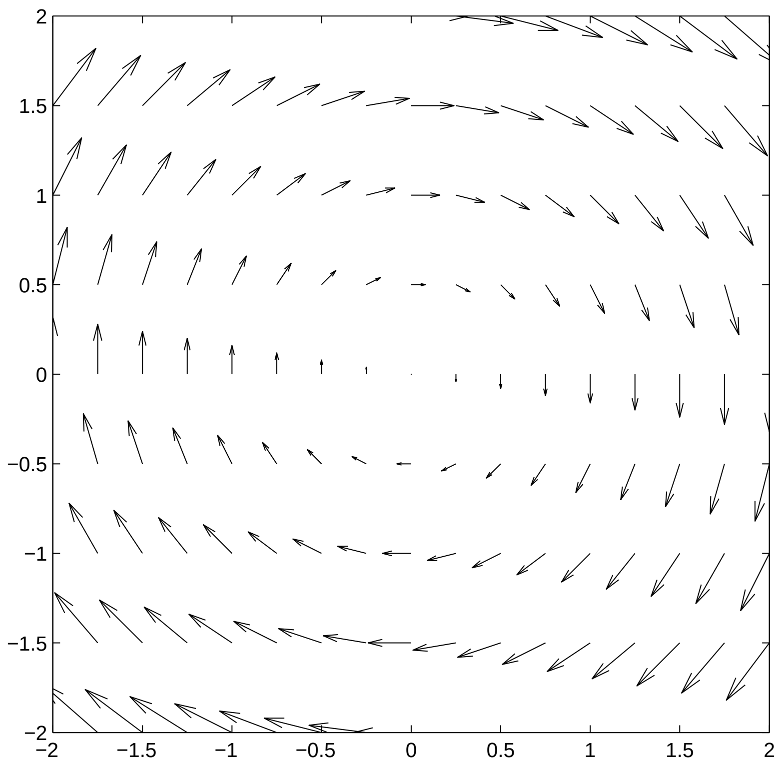
- $\Phi(t) = \mathcal{L}^{-1}\left((sI - A)^{-1}\right)$ is called the *state-transition matrix*; it maps the initial state to the state at time $t$:

$$x(t) = \Phi(t)x(0)$$

(in particular, state $x(t)$ is a linear function of initial state $x(0)$)

# Example 1: Harmonic oscillator

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x$$



$sI - A = \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}$, so resolvent is

$$(sI - A)^{-1} = \begin{bmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ \frac{-1}{s^2+1} & \frac{s}{s^2+1} \end{bmatrix}$$

(eigenvalues are $\pm j$)

state transition matrix is

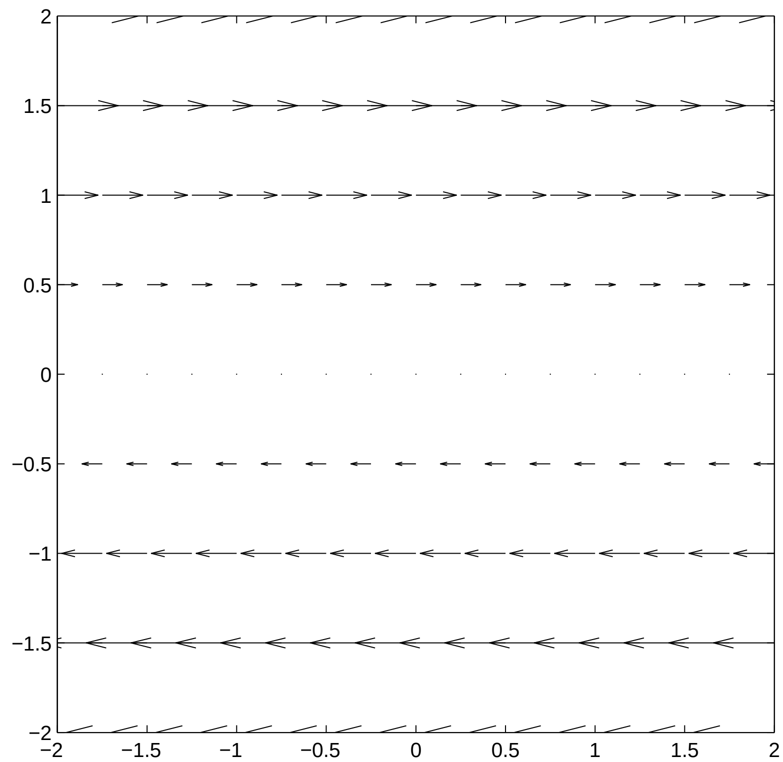
$$\Phi(t) = \mathcal{L}^{-1}\left( \begin{bmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ \frac{-1}{s^2+1} & \frac{s}{s^2+1} \end{bmatrix} \right) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

a rotation matrix ($-t$ radians)

so we have $x(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} x(0)$

# Example 2: Double integrator

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x$$



$sI - A = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}$, so resolvent is

$$(sI - A)^{-1} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix}$$

(eigenvalues are $0$, $0$)

state transition matrix is

$$\Phi(t) = \mathcal{L}^{-1}\left(\begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix}\right) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

so we have $x(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x(0)$

# Characteristic polynomial

$\mathcal{X}(s) = \det(sI - A)$ is called the *characteristic polynomial* of $A$

- $\mathcal{X}(s)$ is a polynomial of degree $n$, with leading ($i.e.$, $s^n$) coefficient one
- roots of $\mathcal{X}$ are the eigenvalues of $A$
- $\mathcal{X}$ has real coefficients, so eigenvalues are either real or occur in conjugate pairs
- there are $n$ eigenvalues (if we count multiplicity as roots of $\mathcal{X}$)

# Eigenvalues of $A$ and poles of resolvent

$i, j$ entry of resolvent can be expressed via Cramer's rule as

$$(-1)^{i+j}\frac{\det \Delta_{ij}}{\det(sI - A)}$$

where $\Delta_{ij}$ is $sI - A$ with $j$th row and $i$th column deleted

- $\det \Delta_{ij}$ is a polynomial of degree less than $n$, so $i, j$ entry of resolvent has form $f_{ij}(s)/\mathcal{X}(s)$ where $f_{ij}$ is polynomial with degree less than $n$

- poles of entries of resolvent must be eigenvalues of $A$

- but not all eigenvalues of $A$ show up as poles of each entry (when there are cancellations between $\det \Delta_{ij}$ and $\mathcal{X}(s)$)

# Matrix exponential

$(I - C)^{-1} = I + C + C^2 + C^3 + \cdots$ (if series converges)

- series expansion of resolvent:

$$(sI - A)^{-1} = (1/s)(I - A/s)^{-1} = \frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \cdots$$

(valid for $|s|$ large enough) so

$$\Phi(t) = \mathcal{L}^{-1}\left((sI - A)^{-1}\right) = I + tA + \frac{(tA)^2}{2!} + \cdots$$

- looks like ordinary power series

$$e^{at} = 1 + ta + \frac{(ta)^2}{2!} + \cdots$$

with square matrices instead of scalars . . .

- define **matrix exponential** as

$$e^M = I + M + \frac{M^2}{2!} + \cdots$$

for $M \in \mathbf{R}^{n \times n}$ (which in fact converges for all $M$)

- with this definition, state-transition matrix is

$$\Phi(t) = \mathcal{L}^{-1}\left((sI - A)^{-1}\right) = e^{tA}$$

# Matrix exponential solution of autonomous LDS

solution of $\dot{x} = Ax$, with $A \in \mathbf{R}^{n \times n}$ and constant, is

$$x(t) = e^{tA}x(0)$$

generalizes scalar case: solution of $\dot{x} = ax$, with $a \in \mathbf{R}$ and constant, is

$$x(t) = e^{ta}x(0)$$

- matrix exponential is *meant* to look like scalar exponential
- some things you'd guess hold for the matrix exponential (by analogy with the scalar exponential) do in fact hold
- but **many things you'd guess are wrong**

**example:** you might guess that $e^{A+B} = e^A e^B$, but it's false (in general)

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$e^A = \begin{bmatrix} 0.54 & 0.84 \\ -0.84 & 0.54 \end{bmatrix}, \qquad e^B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$e^{A+B} = \begin{bmatrix} 0.16 & 1.40 \\ -0.70 & 0.16 \end{bmatrix} \neq e^A e^B = \begin{bmatrix} 0.54 & 1.38 \\ -0.84 & -0.30 \end{bmatrix}$$

however, we do have $e^{A+B} = e^A e^B$ if $AB = BA$, $i.e.$, $A$ and $B$ commute

thus for $t,\ s \in \mathbf{R}$, $e^{(tA+sA)} = e^{tA}e^{sA}$

with $s = -t$ we get

$$e^{tA}e^{-tA} = e^{tA-tA} = e^0 = I$$

so $e^{tA}$ is nonsingular, with inverse

$$\left(e^{tA}\right)^{-1} = e^{-tA}$$

**example:** let's find $e^A$, where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

we already found

$$e^{tA} = \mathcal{L}^{-1}(sI - A)^{-1} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

so, plugging in $t = 1$, we get $e^A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

let's check power series:

$$e^A = I + A + \frac{A^2}{2!} + \cdots = I + A$$

since $A^2 = A^3 = \cdots = 0$

# Time transfer property

for $\dot{x} = Ax$ we know

$$x(t) = \Phi(t)x(0) = e^{tA}x(0)$$

**interpretation:** the matrix $e^{tA}$ propagates initial condition into state at time $t$

more generally we have, for *any* $t$ and $\tau$,

$$x(\tau + t) = e^{tA}x(\tau)$$

(to see this, apply result above to $z(t) = x(t + \tau)$)

**interpretation:** the matrix $e^{tA}$ propagates state $t$ seconds forward in time (backward if $t < 0$)

- recall first order (forward Euler) *approximate* state update, for small $t$:

$$x(\tau + t) \approx x(\tau) + t\dot{x}(\tau) = (I + tA)x(\tau)$$

- *exact* solution is

$$x(\tau + t) = e^{tA}x(\tau) = (I + tA + (tA)^2/2! + \cdots)x(\tau)$$

- forward Euler is just first two terms in series

# Sampling a continuous-time system

suppose $\dot{x} = Ax$

sample $x$ at times $t_1 \leq t_2 \leq \cdots$: define $z(k) = x(t_k)$

then $z(k+1) = e^{(t_{k+1}-t_k)A} z(k)$

for uniform sampling $t_{k+1} - t_k = h$, so

$$z(k+1) = e^{hA} z(k),$$

a discrete-time LDS (called *discretized version* of continuous-time system)

# Piecewise constant system

consider *time-varying* LDS $\dot{x} = A(t)x$, with

$$A(t) = \begin{cases} A_0 & 0 \leq t < t_1 \\ A_1 & t_1 \leq t < t_2 \\ \vdots \end{cases}$$

where $0 < t_1 < t_2 < \cdots$ (sometimes called jump linear system)

for $t \in [t_i, t_{i+1}]$ we have

$$x(t) = e^{(t-t_i)A_i} \cdots e^{(t_3-t_2)A_2} e^{(t_2-t_1)A_1} e^{t_1 A_0} x(0)$$

(matrix on righthand side is called state transition matrix for system, and denoted $\Phi(t)$)

# Qualitative behavior of $x(t)$

suppose $\dot{x} = Ax$, $x(t) \in \mathbf{R}^n$

then $x(t) = e^{tA}x(0)$; $X(s) = (sI - A)^{-1}x(0)$

$i$th component $X_i(s)$ has form

$$X_i(s) = \frac{a_i(s)}{\mathcal{X}(s)}$$

where $a_i$ is a polynomial of degree $< n$

thus the poles of $X_i$ are all eigenvalues of $A$ (but not necessarily the other way around) first assume eigenvalues $\lambda_i$ are distinct, so $X_i(s)$ cannot have repeated poles

then $x_i(t)$ has form

$$x_i(t) = \sum_{j=1}^{n} \beta_{ij} e^{\lambda_j t}$$

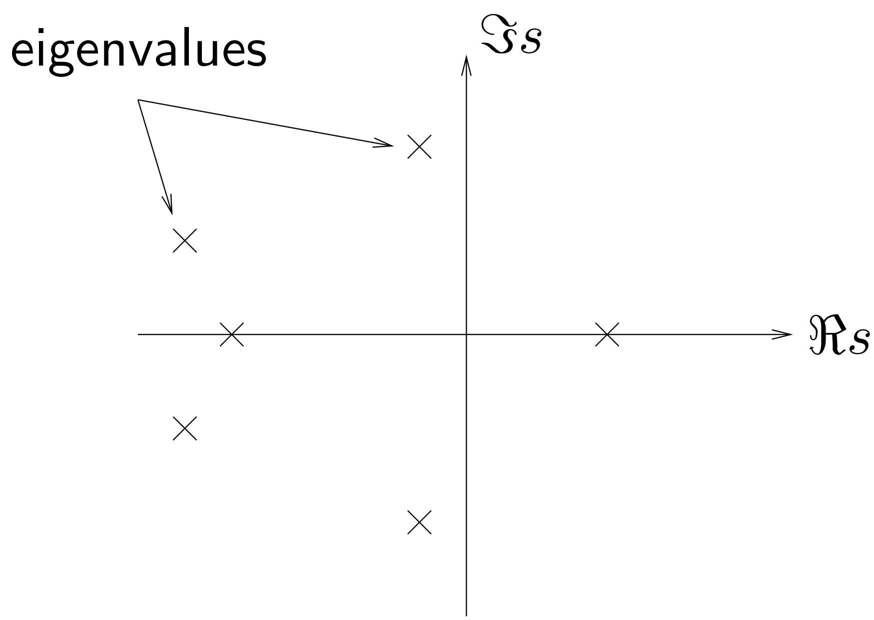
where $\beta_{ij}$ depend on $x(0)$ (linearly)

eigenvalues determine (possible) qualitative behavior of $x$:

- eigenvalues give exponents that can occur in exponentials
- real eigenvalue $\lambda$ corresponds to an exponentially decaying or growing term $e^{\lambda t}$ in solution
- complex eigenvalue $\lambda = \sigma + j\omega$ corresponds to decaying or growing sinusoidal term $e^{\sigma t}\cos(\omega t + \phi)$ in solution

- $\Re\lambda_j$ gives exponential growth rate (if $> 0$), or exponential decay rate (if $< 0$) of term

- $\Im\lambda_j$ gives frequency of oscillatory term (if $\neq 0$)



now suppose $A$ has repeated eigenvalues, so $X_i$ can have repeated poles

express eigenvalues as $\lambda_1, \ldots, \lambda_r$ (distinct) with multiplicities $n_1, \ldots, n_r$, respectively $(n_1 + \cdots + n_r = n)$

then $x_i(t)$ has form

$$x_i(t) = \sum_{j=1}^{r} p_{ij}(t) e^{\lambda_j t}$$

where $p_{ij}(t)$ is a polynomial of degree $< n_j$ (that depends linearly on $x(0)$)

# Stability

we say system $\dot{x} = Ax$ is *stable* if $e^{tA} \to 0$ as $t \to \infty$

**meaning:**

- state $x(t)$ converges to $0$, as $t \to \infty$, no matter what $x(0)$ is
- all trajectories of $\dot{x} = Ax$ converge to $0$ as $t \to \infty$

**fact:** $\dot{x} = Ax$ is stable if and only if all eigenvalues of $A$ have negative real part:

$$\Re\lambda_i < 0, \quad i = 1, \ldots, n$$

the ‘if’ part is clear since

$$\lim_{t\to\infty} p(t)e^{\lambda t} = 0$$

for any polynomial, if $\Re\lambda < 0$

we’ll see the ‘only if’ part next lecture

more generally, $\max_i \Re\lambda_i$ determines the maximum asymptotic logarithmic growth rate of $x(t)$ (or decay, if $< 0$)