Matrix Exponential

Ogata 9-4. Solving the time invariant state equations.

1. $e^{at} = \sum_{k=0}^{\infty} \frac{a^k t^k}{k!} = 1 + at + \frac{1}{2}a^2t^2 + \cdots + \frac{a^k t^k}{k!} + \cdots$

2. $e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = I + At + \frac{1}{2}A^2t^2 + \cdots + \frac{A^k t^k}{k!} + \cdots$

3. matrix exponential 성질

1) $\Phi(t) = e^{At}$ $\qquad e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$

2) $\Phi(-t) = e^{-At}$

3) $\Phi(0) = e^{A0} = I$

4) $\cancel{[\Phi(t)]^{-1} = [e^{At}]^{-1} = [[e^{-At}]^{-1}]^{-1} = [\Phi(-t)^{-1}]^{-1}}$

$\Phi(t) = e^{At} = [e^{-At}]^{-1} = \Phi(-t)^{-1} = \Phi^{-1}(t) = \Phi(-t)$

5) $\Phi(t_1 + t_2) = \Phi(t_1)\Phi(t_2)$

6) $\Phi(nt) = \Phi(t)^n$

예제 9-5, $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$\Phi(t) = ?$ $\Phi(t) = e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{s(s+3)+2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} = \cancel{\begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+3)} \\ \frac{-2}{s(s+3)} & \frac{1}{s+3} \end{bmatrix}}$$

$$= \frac{1}{s^2+3s+2}\begin{bmatrix} & \\ & \end{bmatrix} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$\xrightarrow{\mathcal{L}^{-1}\{\ \}}$

non homogeneous.

$\dot{x} = Ax + Bu.$ $\dot{x} = ax + bu.$

$x(t) = e^{at}x(0) + e^{at}\int_0^t e^{a\tau} b u(\tau) d\tau$

$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} b u(\tau) d\tau$ 비슷.

$= \Phi(t)x(0) + \int \Phi(t-\tau) b u(\tau) d\tau.$

Laplace form

$sX - x(0) = AX + BU.$ $(sI - A)X = x(0) + BU$

$X = (sI-A)^{-1}x(0) + (sI-A)^{-1}BU$

$= \mathcal{L}\{e^{At}\}x(0) + \mathcal{L}\{e^{At}\}BU.$

↓ inverse laplace $e^{At} * u(\tau)$

$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} * Bu(\tau) d\tau$

convolution $f(t) * g(t) = \int_0^t f(t-\tau) g(\tau) d\tau$

$e^{At} * bu(t) = \int_0^t e^{A(t-\tau)} bu(\tau) d\tau.$

Controllability Matrix 유도.

Ogata 9-6 . Controllability.

기본식 :

$$\dot{x} = Ax + Bu. \qquad x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau) \times d\tau$$

가정 및 controllability

$t : t_0 \rightarrow t_1$ State가 $x(t_0) \rightarrow x(t_1)$ 로 가 ㄱ, ㄴ ~~가능한 u.~~

~~가 있음.~~ ~~$x(t_0)$ 가 있음~~

$x(t_0=0) \longrightarrow x(t_1)=0$ 로 가정

$$x(t_1) = e^{At_1}x(\overset{0}{\cancel{t_0}}) + \int_0^{t_1} e^{A(t_1-\tau)}B\,u(\tau)\,d\tau$$

$$= \underset{e^{At_1}x(0)}{\emptyset} + \int_0^{t_1} e^{A(t_1-\tau)}B\,u(\tau)\,d\tau = 0$$

$\cancel{e^{A(t_1-\tau)}}$

$$x(0) = -e^{-At_1}\int(\quad) = -\int e^{-A\tau}B\,u(\tau)\,d\tau.$$

$$e^{-A\tau} = \sum_{k=0} \frac{A^k\,{}^k}{k!} = \sum_{k=0}^{\infty} \alpha(\tau)^k A^k$$

$$x(0) = -\int \sum \alpha^k(\tau) A^k B\,u(\tau)\,d\tau$$

$$= -\sum_{k=0}^{\infty} \underbrace{A^k B}_{\text{Controllability matrix}} \int_0^{t_1} \alpha^k(\tau)\,u(\tau)\,d\tau$$

Observability Matrix

forced, unforced system 동일

controllability와 같은 t 사용

Observable : $y(t)$를 관찰하면 $x(t)$를 알 수 있다.

$\dot{x} = Ax + Bu$

$y = Cx$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$$y = Ce^{At}x(0) + C\int \text{———}$$

관심있는 항목은 초기값인 $x(0)$. A, B, C, u, D는

알고 있는 값이므로

$y = Ce^{At}x(0)$ 만 관심

$$e^{At} = \cancel{\alpha}\sum_{k=0}^{\infty}\frac{A^k t^k}{k!} = \sum_{k=0}^{\infty}\alpha(t)A^k$$

$$y = \sum_k C\alpha(t)A^k x(0) = \cancel{CA^k \Sigma}$$

$$= C\alpha(t)x(0) + CA\alpha(t)x.$$

$$\rightarrow \sum_{k=0}^{\infty} \underline{CA^k} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix}$$