Phase Margin & Gain Margin

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Stability Margins

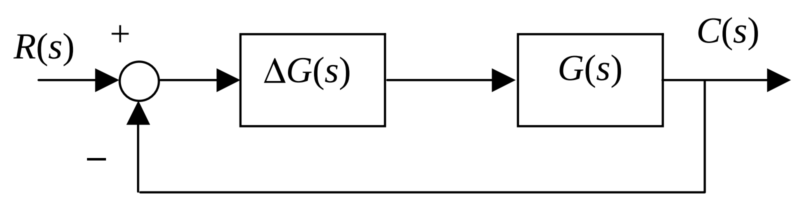
- What is the worst perturbation of the transfer function that will make the system marginally stable?
- Marginal stability for open loop stable systems is when the contour goes through the point (−1,0).

Gain and Phase Margins

Gain Margin: gain perturbation that makes the system marginally stable.

Phase Margin: negative phase perturbation that makes the system marginally stable.

Model Perturbation



- $\Delta G(s)$ = model perturbation
- Gain Margin: $\Delta G(s) = \Delta K$ (gain perturbation)
- Phase Margin: $\Delta G(s) = e^{-j\Delta\theta}$ (phase lag perturbation)

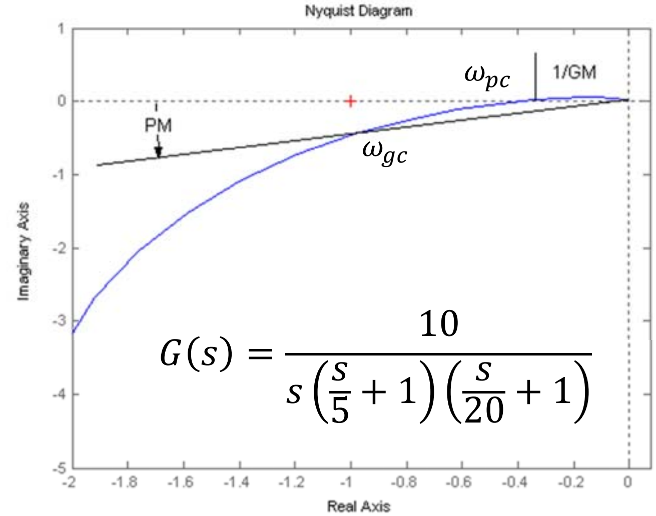
Definitions of Margins

Gain Margin: additional gain that makes the system on the verge of instability.

Phase Margin: additional phase *lag* that makes the system on the verge of instability.

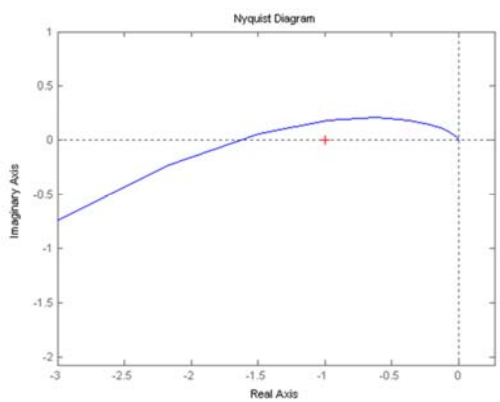
Margins on Polar Plot

ω_{gc}/ω_{pc}=gain crossover ($M = 1$) /phase crossover ($\phi = -180°$)



$$G(s) = \frac{10}{s\left(\frac{s}{5}+1\right)\left(\frac{s}{20}+1\right)}$$

Polar plot of Unstable System

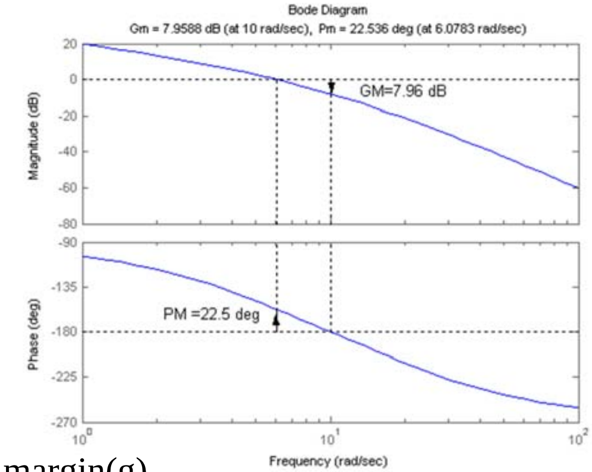


Negative GM (dBs) and PM.

$(-1,0)$ Point on Bode Plot

- Negative real axis for Nyquist plot corresponds to an angle of $-180°$.
- Magnitude of unity corresponds to zero dB.
- For PM we need to find the phase angle at magnitude unity (0dB).
- For GM we need to find the magnitude at an angle of $-180°$.

Bode Plot of Stable System



>> margin(g)

$PM = 180° + \angle G(j\omega_{gc})$, $\quad GM = 1/|G(j\omega_{pc})|$

MATLAB Margin

```
>> [Gm,Pm,Wcg,Wcp] = margin(g)
Gm =
   2.5000
Pm =
  22.5359
Wcg =
  10.0000 (phase crossover freq.)
Wcp =
   6.0783 (gain crossover freq.)
```

$$G(s) = \frac{10}{s\left(\frac{s}{5}+1\right)\left(\frac{s}{20}+1\right)}$$

Gain Margin Calculation

- Multiply numerator and denominator by the complex conjugate of the denominator.

$$G(j\omega) = \frac{N(j\omega)}{D(j\omega)} \times \frac{D^*(j\omega)}{D^*(j\omega)} = \frac{N(j\omega)D^*(j\omega)}{|D(j\omega)|^2}$$

- Equate the imaginary part of the numerator to zero and solve for the phase crossover frequency: $Im[N(j\omega_{pc})D^*(j\omega_{pc})] = 0$
- Calculate the gain margin $GM = -1/G(j\omega_{pc})$

Example: Gain Margin

- Solve for phase crossover (imaginary part zero)

$$G(j\omega) = \frac{1000}{j\omega(j\omega+5)(j\omega+20)}$$

$$\frac{G(j\omega)}{1000} = \frac{-j(-j\omega+5)(-j\omega+20)}{\omega(\omega^2+25)(\omega^2+400)}$$

$$\text{Im}\{G(j\omega)\} = 0$$

$$\Leftrightarrow \text{Re}\{(-j\omega+5)(-j\omega+20)\} = 0$$

$$\Leftrightarrow 100 - \omega^2 = 0 \Leftrightarrow \omega_{pc} = 10\, rad/s$$

Calculate Gain Margin

- Evaluate the magnitude at the phase crossover frequency

$$GM = -\frac{1}{G(j10)}$$
$$= -\frac{(j10)(j10+5)(j10+20)}{1000} = 2.5$$

Phase Margin Calculation

- Solve for gain crossover frequency (unity magnitude)

$$\left|G(j\omega_{gc})\right|^2 = 1 \Leftrightarrow \left|N(j\omega_{gc})\right|^2 = \left|D(j\omega_{gc})\right|^2$$

- Calculate the phase margin

$$PM = 180° + \angle G(j\omega_{gc})$$

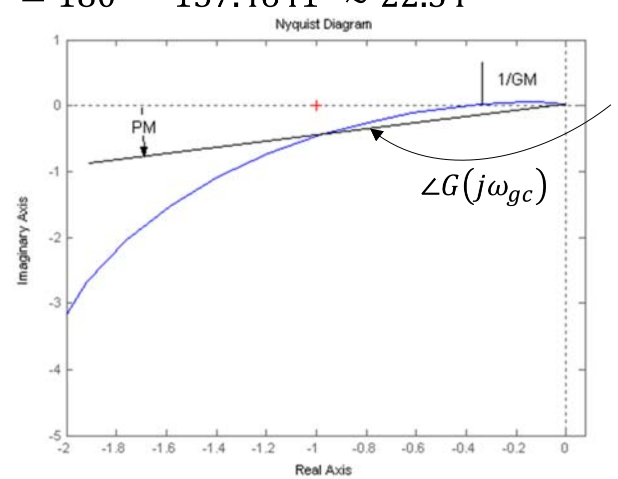
Phase Margin Calculation

- Solve for gain crossover (unity magnitude)

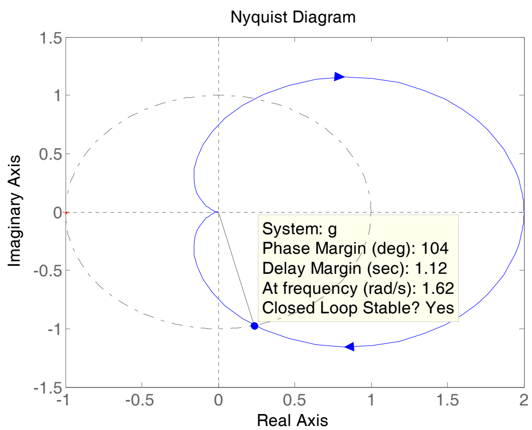
$$G(j\omega) = \frac{1000}{j\omega(j\omega+5)(j\omega+20)}$$
$$|G(j\omega)|^2 = \frac{10^6}{\omega^2(\omega^2+25)(\omega^2+400)} = 1$$
$$\Leftrightarrow \omega^6 + 425\omega^4 + 10^4\omega^2 - 10^6 = 0$$
$$\omega^2 = -393.0887, \qquad -68.8586, \qquad 39.9456$$
$$\omega_{gc} = 6.078\, rad/s$$

Phase Margin

$$PM = 180° + \angle G(j\omega_{gc}) = 180° + \angle G(j6.078)$$
$$= 180° - 157.4641° \approx 22.54$$



Stability Margins




```
>> g=zpk([],[-1,-5],10)

Zero/pole/gain:
    10
-----------
(s+1) (s+5)

>> nyquist(g)
```


Delay Margin

T_{dm}= delay margin = time delay for the system to be on the verge of instability.

- Transfer function with time delay T_d

$$G(s)e^{-sT_d}$$

- System on verge of instability

$$G(j\omega_{gc})e^{-j\omega_{gc}T_{dm}} = -1$$

Delay Margin Calculation

$$G(j\omega_{gc})e^{-j\omega_{gc}T_{dm}} = -1$$

- Equate angles:

$$\angle G(j\omega_{gc}) - \omega_{gc}T_{dm} \times \frac{180°}{\pi} = -180°$$

$$\angle G(j\omega_{gc}) = -180° + \text{PM}$$

- Solve for the delay margin

$$T_{dm} = \frac{\text{PM}}{\omega_{gc}} \times \frac{\pi}{180°} = \frac{180° + \angle G(j\omega_{gc})}{\omega_{gc}} \times \frac{\pi}{180°}$$

Example

$$PM = 180° + \angle G(j\omega_{gc}) = 104°$$

$$\omega_{gc} = 1.62\ rad/s$$

$$T_{dm} = \frac{PM}{\omega_{gc}} \times \frac{\pi}{180°}$$

$$= \frac{104°}{1.62} \times \frac{\pi}{180°} = 1.12$$

Same as MATLAB answer.