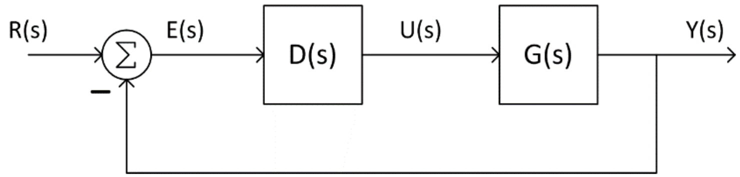
SECTION 4: STEADY-STATE ERROR

ESE 430 – Feedback Control Systems

Introduction

Steady-State Error – Introduction

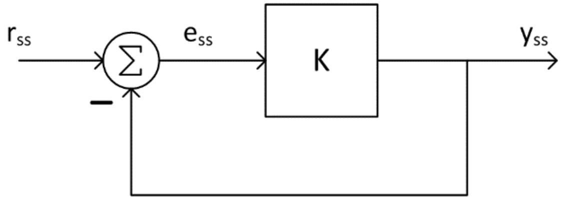
- Consider a simple ***unity-feedback system***



- The ***error*** is the difference between the reference and the output

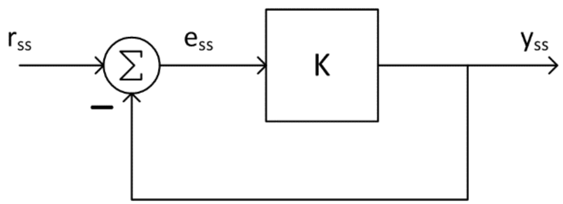
$$E(s) = R(s) - Y(s)$$

  - The input to the controller, $D(s)$
- Consider a case where:
  - Reference input is a step
  - Plant has no poles at the origin – finite DC gain
  - Controller is a simple gain block
- In ***steady state***, the forward path reduces to a constant gain:



Steady-State Error – Introduction

- In steady state, we'd like:
  - Output to be equal to the input: $y_{ss} = r_{ss}$
  - Zero steady-state error: $e_{ss} = 0$
- Is that the case here?



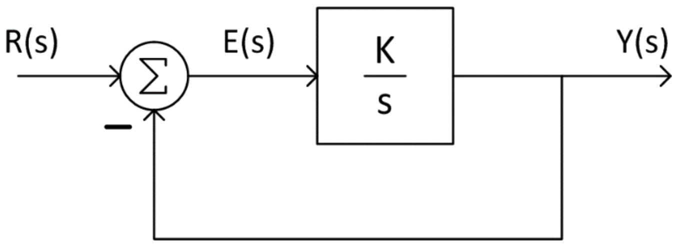
$$e_{ss} = r_{ss} - y_{ss} = r_{ss} - e_{ss}K$$

$$e_{ss} = r_{ss}\frac{1}{1+K}$$

  - ***No***, if $r_{ss} \neq 0$, then $e_{ss} \neq 0$
- ***Non-zero steady-state error to a step input for finite steady-state forward-path gain***
  - Finite DC gain implies ***no poles at the origin*** in $D(s)$ or $G(s)$

Steady-State Error – Introduction

- Now, allow a single pole at the origin
  - An ***integrator*** in the forward path



- Now the error is

$$E(s) = R(s) - E(s) \cdot \frac{K}{s}$$

$$E(s) = R(S)\frac{s}{s+K}$$

- For a step input

$$E(s) = \frac{1}{s}\frac{s}{s+K} = \frac{1}{s+K}$$

- Applying the final value theorem gives the steady-state error

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0}\frac{s}{s+K} = 0$$

- ***Zero steady-state error to a step input when there is an integrator in the forward path***

Steady-State Error – Introduction

- Next, consider a ramp input to the same system

$$r(t) = t \cdot u(t) \;\text{ and }\; R(s) = \frac{1}{s^2}$$

- Now the error is

$$E(s) = \frac{1}{s^2}\frac{s}{s+K} = \frac{1}{s(s+K)}$$

- The steady-state error is

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s}{s(s+K)} = \frac{1}{K}$$

- ***Non-zero, but finite, steady-state error to a ramp input when there is an integrator in the forward path***

Steady-State Error – Introduction

- Two key observations from the preceding example involving ***unity-feedback*** systems:
  - ***Steady-state error is related to the number of integrators in the open-loop transfer function***
  - ***Steady-state error is related to the type of input***
- We'll now explore both of these observations more thoroughly
- First, we'll introduce the concept of ***system type***

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System Type and Steady-State Error

System Type

- **System Type**

  The degree of the input polynomial for which the steady-state error is a finite, non-zero constant

  - **Type 0**: finite, non-zero error to a ***step*** input
  - **Type 1**: finite, non-zero error to a ***ramp*** input
  - **Type 2**: finite, non-zero error to a ***parabolic*** input

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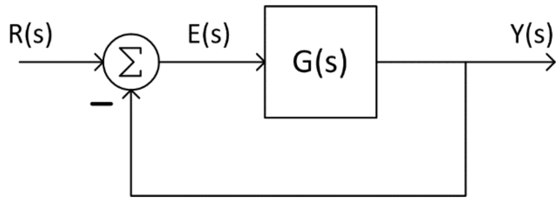
- For the remainder of this sub-section, and the one that follows, we'll consider only the special case of ***unity-feedback*** systems

System Type – Unity-Feedback Systems



- For ***unity-feedback systems***, system type is determined by the ***number of integrators in the forward path***
  - **<u>Type 0</u>**: no integrators in the open-loop TF, e.g.:

$$G(s) = \frac{s+4}{(s+6)(s^2+4s+8)}$$

  - **<u>Type 1</u>**: one integrator in the open-loop TF, e.g.:

$$G(s) = \frac{15}{s(s^2+3s+12)}$$

  - **<u>Type 2</u>**: two integrators in the open-loop TF, e.g.:

$$G(s) = \frac{s+5}{s^2(s+3)(s+7)}$$

Types of Inputs

- When characterizing a control system's error performance we focus on three main inputs:
  - Step
  - Ramp
  - Parabola
- We will derive expressions for the steady-state error due to each

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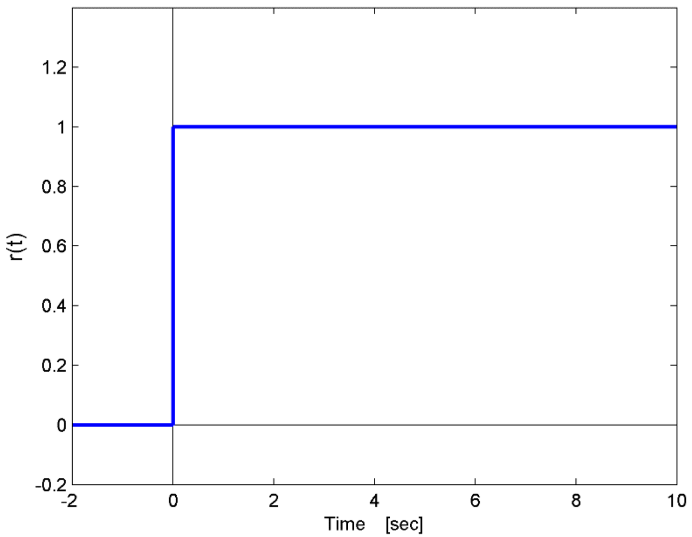
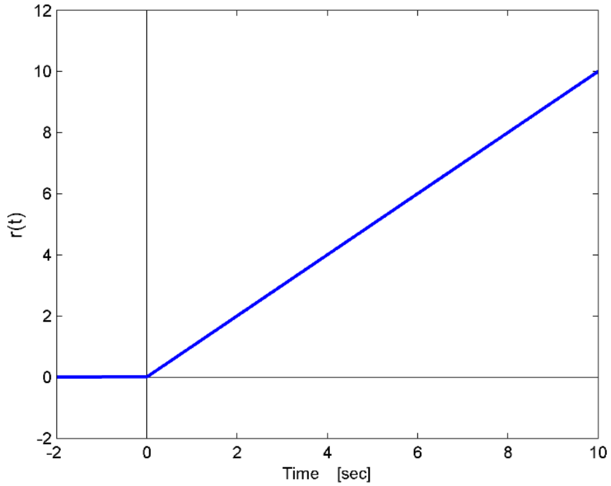
- **Step**:
  - $r(t) = u(t) \leftrightarrow \quad R(s) = \frac{1}{s}$
  - For a positioning system, this represents a ***constant position***

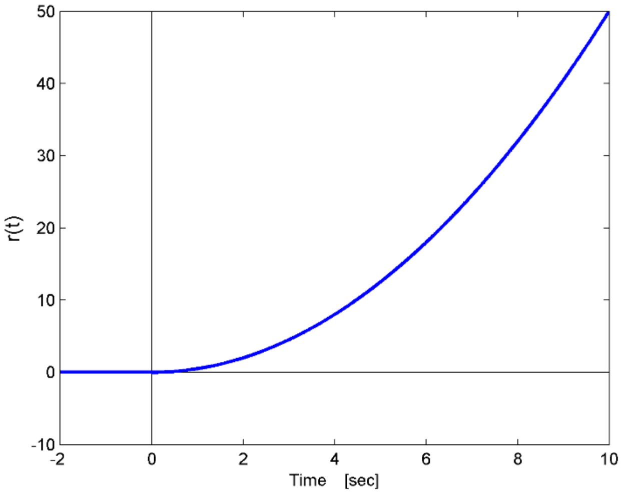


Types of Inputs

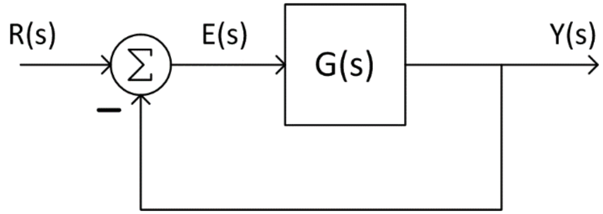
- **Ramp**:
  - $r(t) = t \cdot u(t) \leftrightarrow \quad R(s) = \frac{1}{s^2}$
  - For a positioning system, this represents a ***constant velocity***



- **Parabola**:
  - $r(t) = \frac{1}{2}t^2 \cdot u(t) \leftrightarrow \quad R(s) = \frac{1}{s^3}$
  - For a positioning system, this represents a ***constant acceleration***



Steady-State Error – Unity-Feedback



- For ***unity-feedback systems*** steady-state error can be expressed in terms of the ***open-loop transfer function***, $G(s)$

$$E(s) = R(s) - Y(s) = R(s) - E(s)G(s)$$

$$E(s) = \frac{R(s)}{1 + G(s)}$$

- Steady-state error is found by applying the ***final value theorem***

$$e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}$$

- We'll now consider this expression for each of the three inputs of interest

Steady-State Error – Step Input

- For a step input

$$r(t) = u(t) \quad \leftrightarrow \quad R(s) = \frac{1}{s}$$

- Steady-state error to a step input is

$$e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{s\frac{1}{s}}{1 + G(s)}$$

$$e_{ss} = \lim_{s \to 0} \frac{1}{1 + G(s)}$$

$$\boxed{e_{ss} = \frac{1}{1 + \lim_{s \to 0} G(s)}}$$

Steady-State Error – Step Input

$$e_{ss} = \frac{1}{1 + \lim_{s \to 0} G(s)}$$

- In order to have $e_{ss} = 0$, as we'd like, we must have

$$\lim_{s \to 0} G(s) = \infty$$

- That is, the ***DC gain of the open-loop system must be infinite***
- If $G(s)$ has the following form

$$G(s) = \frac{(s + z_1)(s + z_2)\cdots}{(s + p_1)(s + p_2)\cdots}$$

then

$$\lim_{s \to 0} G(s) = \frac{z_1 z_2 \cdots}{p_1 p_2 \cdots} \neq \infty$$

and we'll have non-zero steady-state error

Steady-State Error – Step Input

- However, consider $G(s)$ of the following form

$$G(s) = \frac{(s+z_1)(s+z_2)\cdots}{s^n(s+p_1)(s+p_2)\cdots}$$

where $n \geq 1$

- That is, $G(s)$ includes n integrators
  - It is a ***type n system***

$$\lim_{s\to 0} G(s) = \infty \quad \text{and} \quad e_{ss} = 0$$

- ***A type 1 or greater system will exhibit zero steady-state error to a step input***

Steady-State Error – Ramp Input

- For a ramp input

$$r(t) = t \cdot u(t) \quad \leftrightarrow \quad R(s) = \frac{1}{s^2}$$

- Steady-state error to a ramp input is

$$e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{s\frac{1}{s^2}}{1 + G(s)}$$

$$e_{ss} = \lim_{s \to 0} \frac{1}{s + sG(s)}$$

$$\boxed{e_{ss} = \frac{1}{\lim_{s \to 0} sG(s)}}$$

Steady-State Error – Ramp Input

$$e_{ss} = \frac{1}{\lim_{s\to 0} sG(s)}$$

- In order to have $e_{ss} = 0$, the following must be true

$$\lim_{s\to 0} sG(s) = \infty$$

- If there are no integrators in the forward path, then

$$\lim_{s\to 0} sG(s) = \lim_{s\to 0} s\frac{(s+z_1)(s+z_2)\cdots}{(s+p_1)(s+p_2)\cdots} = 0$$

and

$$e_{ss} = \infty$$

- ***A type 0 system has infinite steady-state error to a ramp input***

Steady-State Error – Ramp Input

- If there is a single integrator in the forward path, i.e. a type 1 system

$$G(s) = \frac{(s+z_1)(s+z_2)\cdots}{s(s+p_1)(s+p_2)\cdots}$$

then

$$\lim_{s\to 0} sG(s) = \frac{z_1 z_2 \cdots}{p_1 p_2 \cdots}$$

and

$$e_{ss} = \frac{p_1 p_2 \cdots}{z_1 z_2 \cdots}$$

- ***A type 1 system has non-zero, but finite, steady-state error to a ramp input***

Steady-State Error – Ramp Input

- If there are two or more integrators in the forward path, i.e. a type 2 or greater system

$$G(s) = \frac{(s+z_1)(s+z_2)\cdots}{s^n(s+p_1)(s+p_2)\cdots}, \quad (n \geq 2)$$

then

$$\lim_{s\to 0} sG(s) = \lim_{s\to 0} s\frac{(s+z_1)(s+z_2)\cdots}{s^n(s+p_1)(s+p_2)\cdots} = \infty$$

and

$$e_{ss} = 0$$

- ***A type 2 or greater system has zero steady-state error to a ramp input***

Steady-State Error – Parabolic Input

- For a Parabolic input

$$r(t) = \frac{t^2}{2} \cdot u(t) \quad \leftrightarrow \quad R(s) = \frac{1}{s^3}$$

- Steady-state error to a parabolic input is

$$e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{s \frac{1}{s^3}}{1 + G(s)}$$

$$e_{ss} = \lim_{s \to 0} \frac{1}{s^2 + s^2 G(s)}$$

$$\boxed{e_{ss} = \frac{1}{\lim_{s \to 0} s^2 G(s)}}$$

Steady-State Error – Parabolic Input

$$e_{ss} = \frac{1}{\lim_{s \to 0} s^2 G(s)}$$

- In order to have $e_{ss} = 0$, the following must be true

$$\lim_{s \to 0} s^2 G(s) = \infty$$

- If there are no integrators in the forward path, then

$$\lim_{s \to 0} s^2 G(s) = \lim_{s \to 0} s^2 \frac{(s + z_1)(s + z_2) \cdots}{(s + p_1)(s + p_2) \cdots} = 0$$

and

$$e_{ss} = \infty$$

- ***A type 0 system has infinite steady-state error to a parabolic input***

Steady-State Error – Parabolic Input

- If there is a single integrator in the forward path, i.e. a type 1 system

$$G(s) = \frac{(s+z_1)(s+z_2)\cdots}{s(s+p_1)(s+p_2)\cdots}$$

then

$$\lim_{s\to 0} s^2 G(s) = s\frac{z_1 z_2 \cdots}{p_1 p_2 \cdots} = 0$$

and

$$e_{ss} = \infty$$

- ***A type 1 system has infinite steady-state error to a parabolic input***

Steady-State Error – Parabolic Input

- If there are two integrators in the forward path, i.e. a type 2 system

$$G(s) = \frac{(s+z_1)(s+z_2)\cdots}{s^2(s+p_1)(s+p_2)\cdots}$$

then

$$\lim_{s\to 0} s^2 G(s) = \lim_{s\to 0} \frac{(s+z_1)(s+z_2)\cdots}{(s+p_1)(s+p_2)\cdots} = \frac{z_1 z_2\cdots}{p_1 p_2\cdots}$$

and

$$e_{ss} = \frac{p_1 p_2\cdots}{z_1 z_2\cdots}$$

- ***A type 2 system has non-zero, but finite, steady-state error to a parabolic input***

Steady-State Error – Parabolic Input

- If there are three or more integrators in the forward path, i.e. a type 3 or greater system

$$G(s) = \frac{(s+z_1)(s+z_2)\cdots}{s^n(s+p_1)(s+p_2)\cdots}, \quad (n \geq 3)$$

then

$$\lim_{s\to 0} s^2 G(s) = \lim_{s\to 0} s^2 \frac{(s+z_1)(s+z_2)\cdots}{s^n(s+p_1)(s+p_2)\cdots} = \infty$$

and

$$e_{ss} = 0$$

- ***A type 3 or greater system has zero steady-state error to a parabolic input***

Static Error Constants

Static Error Constants – Unity-Feedback

- We've seen that the steady-state error to each of the inputs considered is
  - **Step**: $$e_{ss} = \frac{1}{1+\lim_{s\to 0} G(s)}$$
  - **Ramp**: $$e_{ss} = \frac{1}{\lim_{s\to 0} sG(s)}$$
  - **Parabola**: $$e_{ss} = \frac{1}{\lim_{s\to 0} s^2 G(s)}$$
- The limit term in each expression is the ***static error constant*** associated with that particular input:
  - **Position constant**: $$K_p = \lim_{s\to 0} G(s)$$
  - **Velocity constant**: $$K_v = \lim_{s\to 0} sG(s)$$
  - **Acceleration constant**: $$K_a = \lim_{s\to 0} s^2 G(s)$$

Steady-State Error vs. System Type

- Steady-state error vs. input and system type

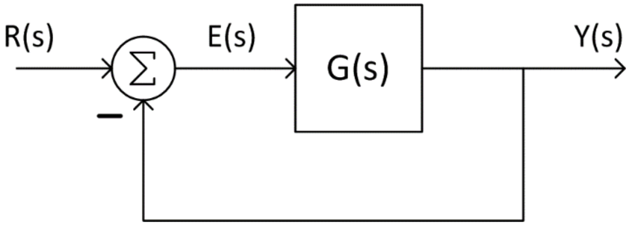
System Type	Input		
	Step	**Ramp**	**Parabola**
0	$\frac{1}{1+K_p}$	∞	∞
1	0	$\frac{1}{Kv}$	∞
2	0	0	$\frac{1}{Ka}$
3	0	0	0

- Note that the given steady-state error is for inputs of ***unit magnitude***
  - Actual error is scaled by the magnitude of the reference input

Non-Unity-Feedback Systems

Non-Unity-Feedback Systems

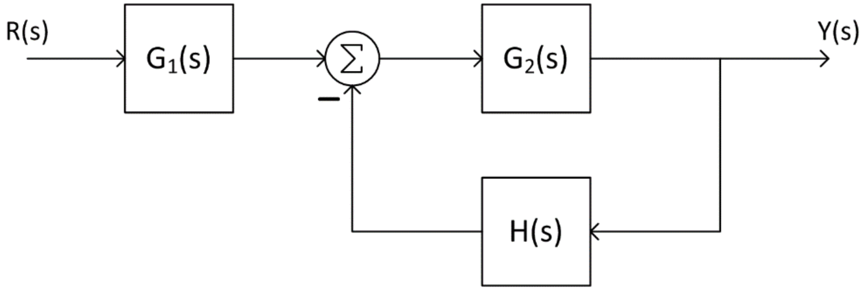
- So far, we've focused on the special case of ***unity-feedback*** systems



  - ***System type*** determined by ***# of integrators*** in the forward path – i.e., # of *open-loop* poles at the origin
  - Steady-state error determined using ***static error constants***
  - Static error constants determined from the ***open-loop transfer function***

Non-Unity-Feedback Systems

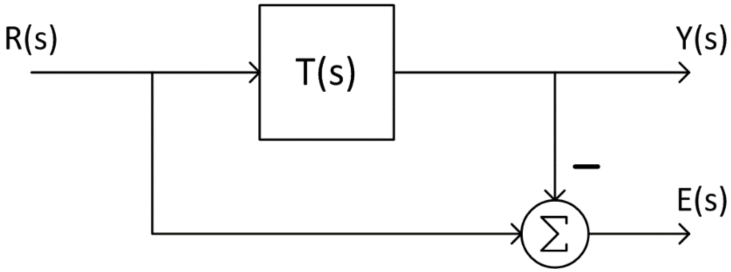
- More general approach to determining steady-state error is to use the ***closed-loop transfer function***
  - Applicable to non-unity-feedback systems, e.g.:



$$T(s) = \frac{G_1(s)G_2(s)}{1 + G_2(s)H(s)}$$

- The error is

$$E(s) = R(s) - Y(s) = R(s) - R(s)T(s)$$



$$E(s) = R(s)[1 - T(s)]$$

Non-Unity-Feedback Systems

- Apply the ***final value theorem*** to determine the steady-state error:

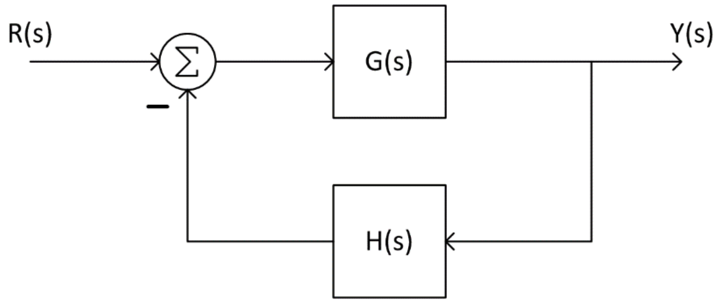
$$e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} sR(s)[1 - T(s)]$$

- Here, system type is determined by using the more general definition:

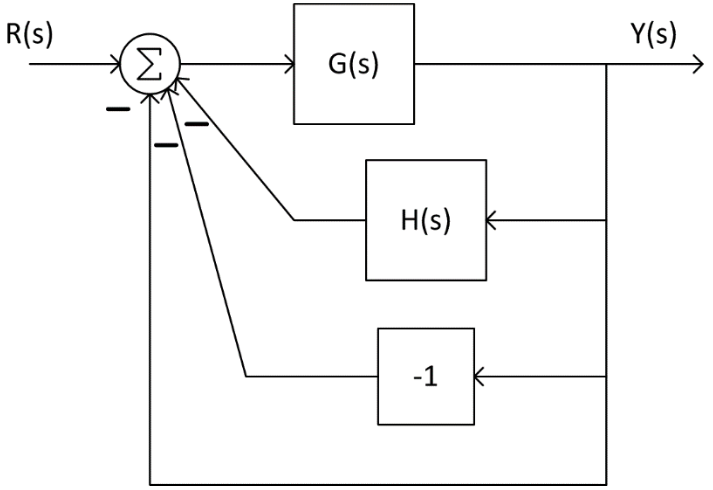
System type *is the degree of the input polynomial for which the steady-state error is a finite, non-zero constant*

Non-Unity-Feedback Systems

- Alternatively, find steady-state error by converting to a unity-feedback configuration, e.g.:

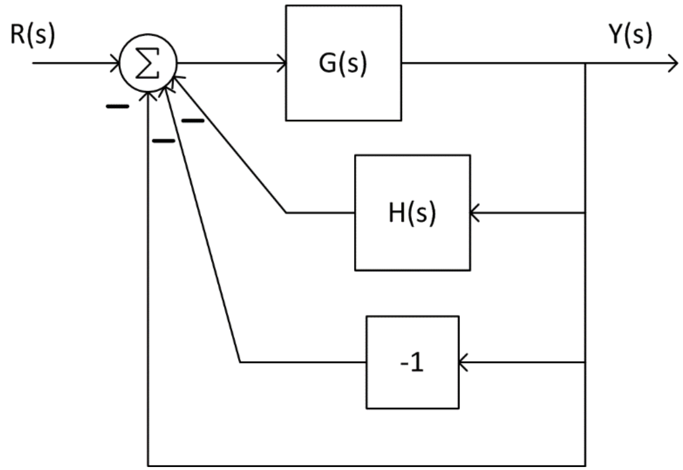


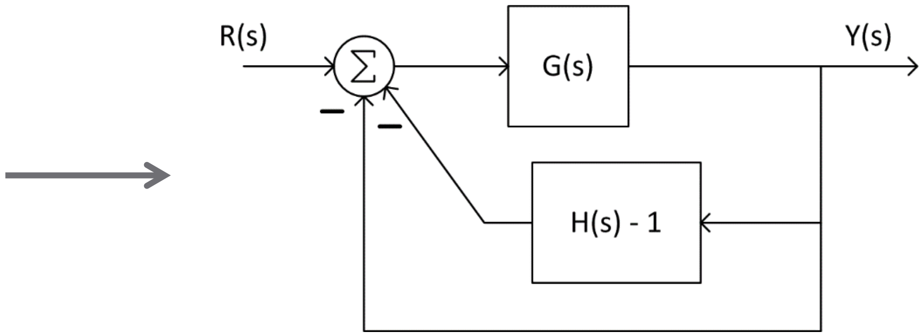
- Add and subtract unity-feedback paths:



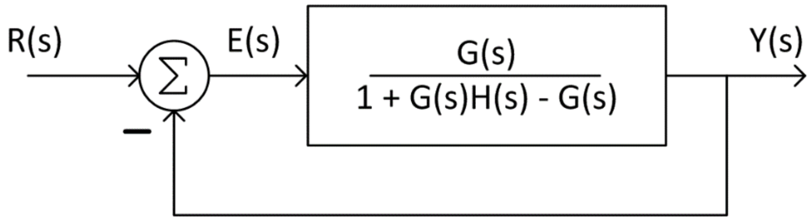
Non-Unity-Feedback Systems

- Combine the two upper parallel feedback paths:





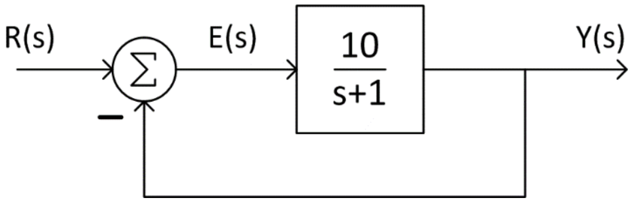
- Collapsing the inner feedback form leaves a unity-feedback system
  - Can now apply unity-feedback error analysis techniques



Steady-State Error – Examples

Steady-State Error – Example 1

- What is the steady-state error to a constant reference input, $r(t) = 3\ cm \cdot u(t)$, for the following feedback positioning system?



- A type 0 system
  - Non-zero error to a constant reference
- Position constant:

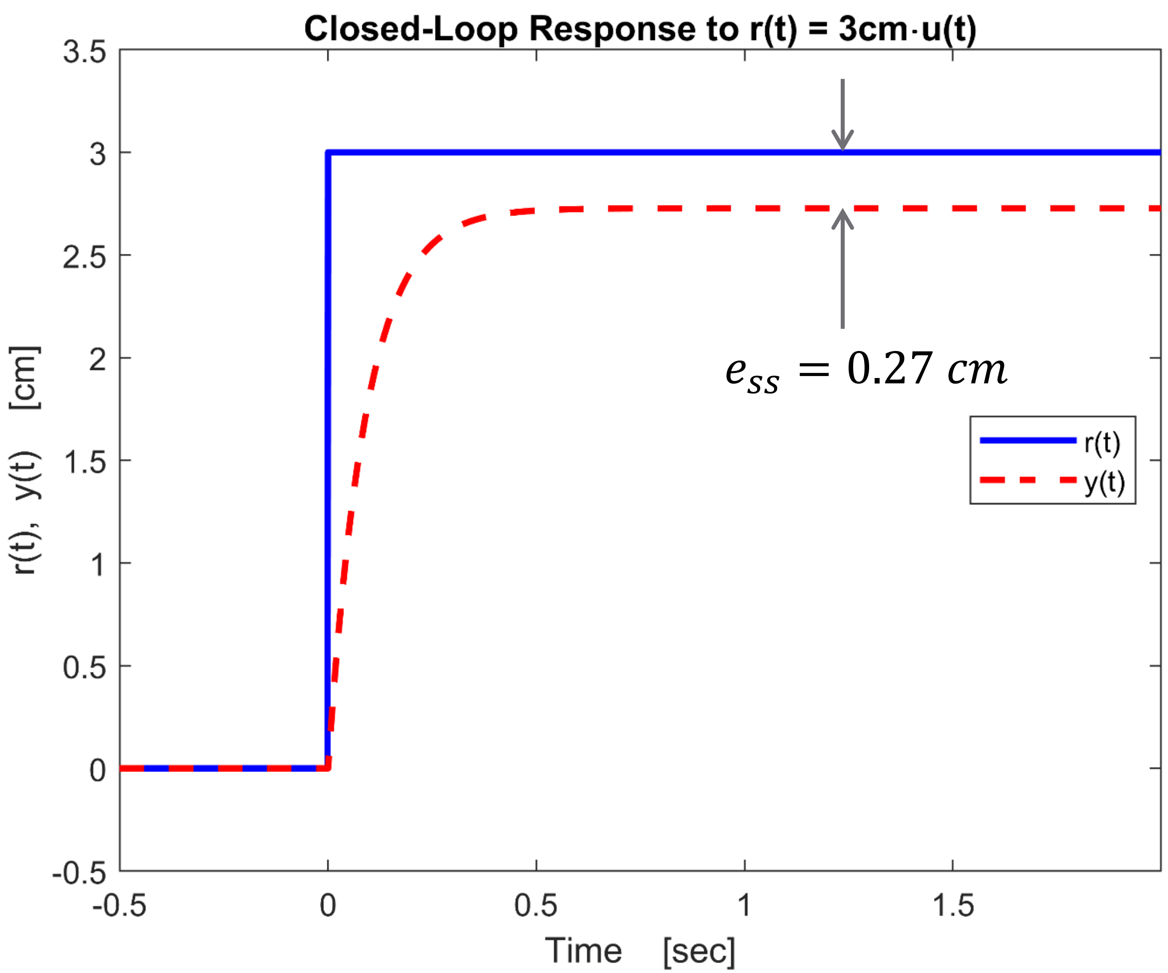
$$K_p = \lim_{s \to 0} G(s) = 10$$

- Steady-state error:

$$e_{ss} = r_{ss} \frac{1}{1 + K_p} = 3\ cm \frac{1}{1 + 10}$$

$$\boxed{e_{ss} = 0.27\ cm}$$

Steady-State Error – Example 1



Steady-State Error – Example 1

- What is the same system's steady-state error to a unit ramp input, $r(t) = t \cdot u(t)$?
  - A type 0 system, so error to a ramp reference will be ***infinite***
- Verify using closed-loop transfer function

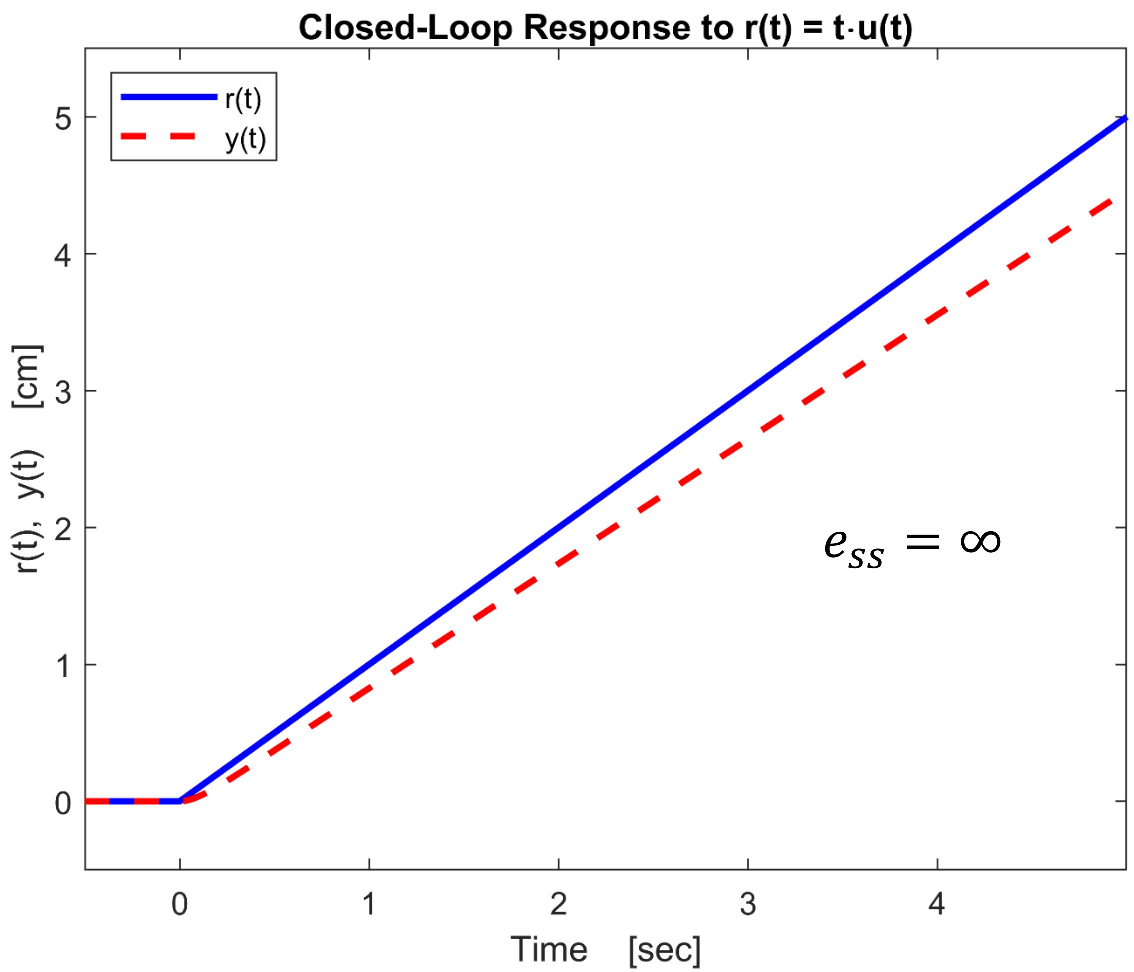
$$T(s) = \frac{G(s)}{1 + G(s)} = \frac{10}{s + 11}$$

- Steady-state error is

$$e_{ss} = \lim_{s \to 0} sR(s)[1 - T(s)] = \lim_{s \to 0} s \frac{1}{s^2} \left[1 - \frac{10}{s + 11}\right]$$

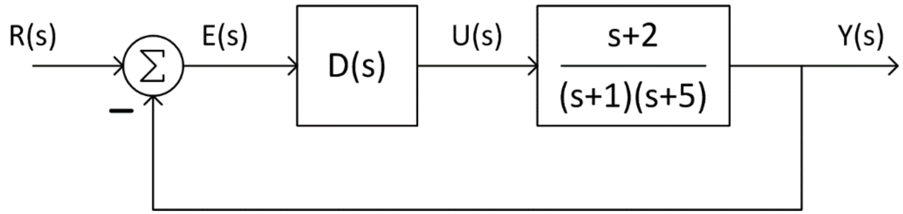
$$e_{ss} = \lim_{s \to 0} \frac{1}{s} \left[\frac{s + 1}{s + 11}\right] = \infty$$

Steady-State Error – Example 1



Steady-State Error – Example 2

- Design the controller, $D(s)$, for error of 0.05 to a unit ramp input



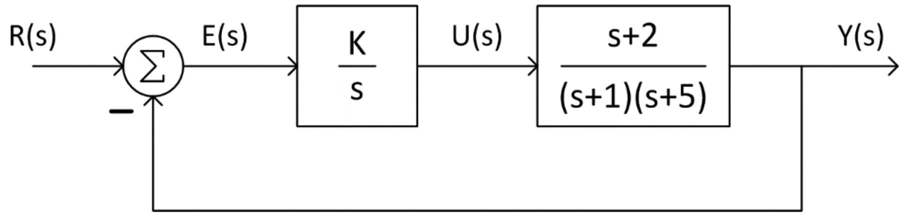
- Plant is type 0
  - Forward path must be type 1 for finite error to a ramp input
  - $D(s)$ must be type 1, so one very simple option is:

$$D(s) = \frac{K}{s}$$

  - Forward-path transfer function is

$$D(s)G(s) = \frac{K(s+2)}{s(s+1)(s+5)}$$

Steady-State Error – Example 2



- The velocity constant is

$$K_v = \lim_{s \to 0} sG(s) = \lim_{s \to 0} s \frac{K(s+2)}{s(s+1)(s+5)} = \frac{2K}{5}$$

- Steady-state error is

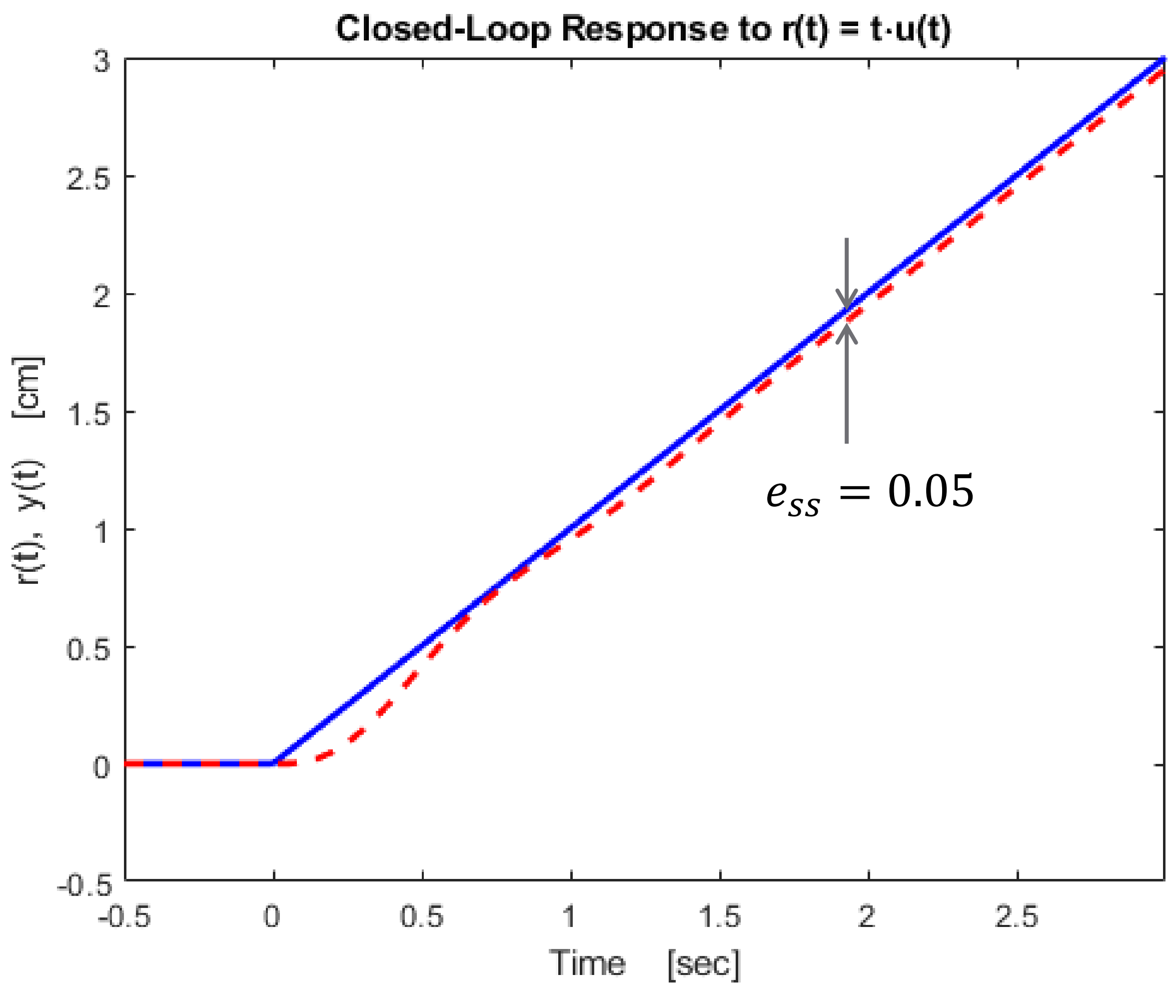
$$e_{ss} = \frac{1}{K_v} = \frac{5}{2K}$$

- For error of 0.05:

$$e_{ss} = 0.05 = \frac{5}{2K}$$

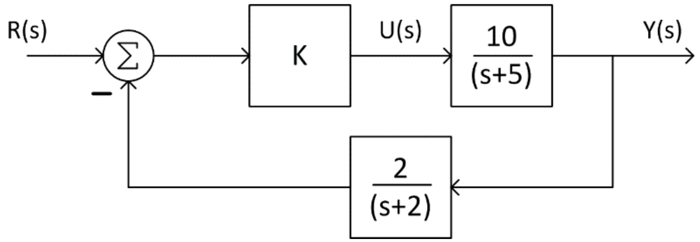
$$K = 50$$

Steady-State Error – Example 2

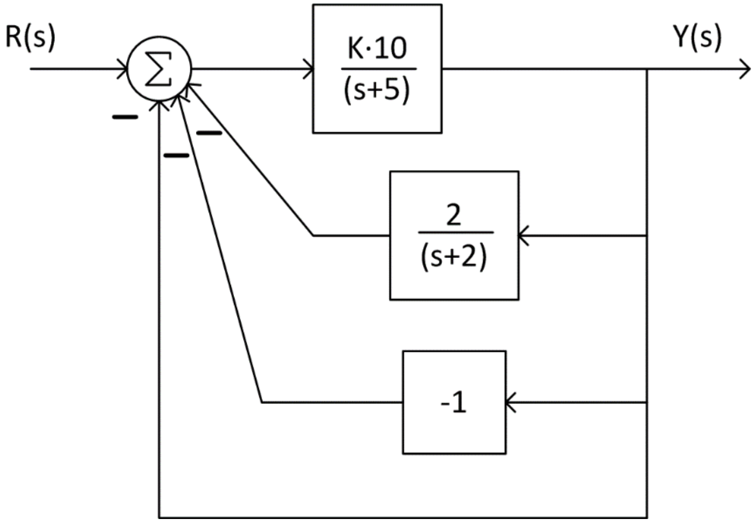


Steady-State Error – Example 3

- Next, consider a non-unity-feedback system:

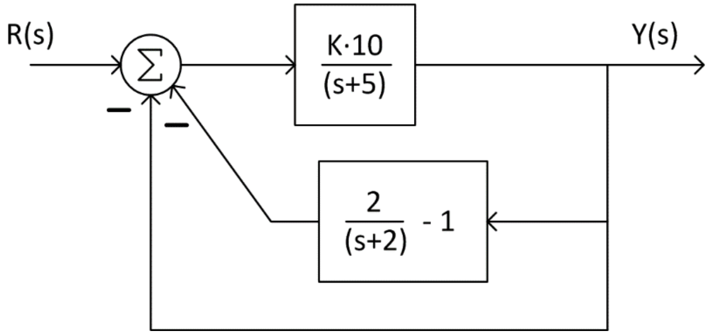


- Determine controller gain, K, to provide a 2% steady-state error to a constant reference input
- First, convert to a unity-feedback system
  - Combine forward-path blocks
  - Simultaneously add and subtract unity-feedback paths

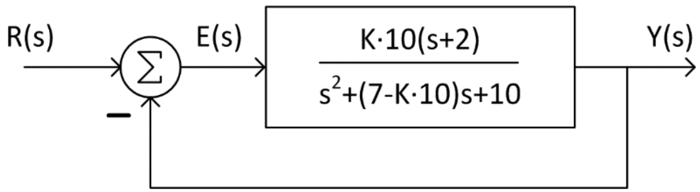


Steady-State Error – Example 3

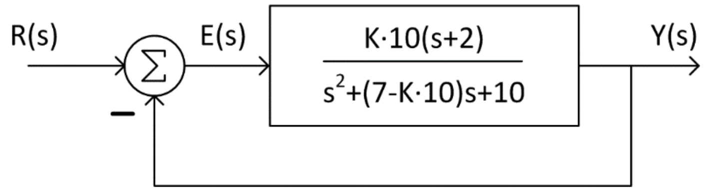
- Combine the top two parallel feedback paths



- Simplifying the inner feedback form leaves a unity-feedback system



Steady-State Error – Example 3



- Steady-state error for this type 0 system is

$$e_{ss} = \frac{1}{1 + K_p}$$

where

$$K_p = \lim_{s \to 0} G(s) = \frac{20 \cdot K}{10} = 2 \cdot K$$

- For 2% steady-state error

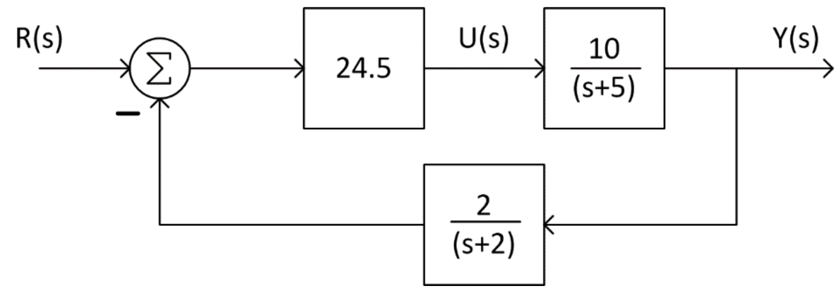
$$e_{ss} = 0.02 = \frac{1}{1 + 2 \cdot K}$$

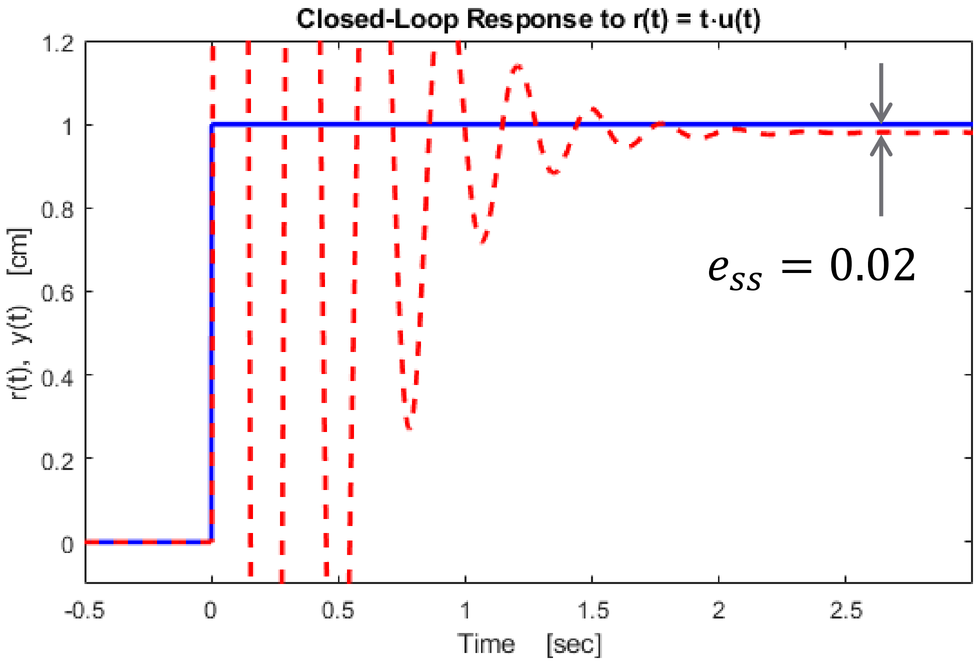
- The controller gain is

$$K = 24.5$$

Steady-State Error – Example 3

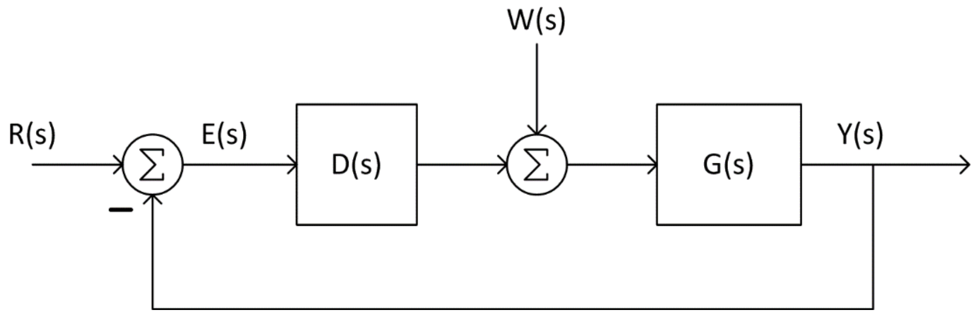
- Note that the controller gain has been set to satisfy a steady-state error requirement *only*
  - Closed loop poles are *very* lightly-damped
  - Dynamic response is likely unacceptable





Steady-State Error – Example 4

- Now, consider a unity-feedback system with a disturbance input

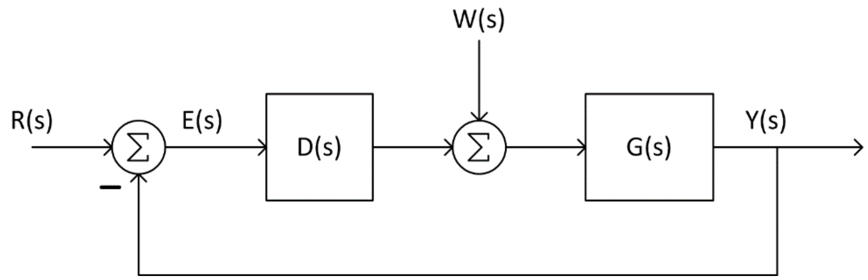


where

$$D(s) = K \quad \text{and} \quad G(s) = \frac{1}{s+5}$$

- Determine the controller gain, K, such that error due to a constant disturbance is 1% of $W(s)$
- For this value of K, what is the steady-state error to a constant reference input?

Steady-State Error – Example 4



- The total error is given by

$$E(s) = R(s) - Y(s) = R(s) - [E(s)D(s)G(s) + W(s)G(s)]$$

$$E(s)[1 + D(s)G(s)] = R(s) - W(s)G(s)$$

$$E(s) = R(s)\frac{1}{1 + D(s)G(s)} - W(s)\frac{G(s)}{1 + D(s)G(s)}$$

- Substituting in controller and plant transfer functions gives

$$E(s) = R(s)\frac{s + 5}{s + 5 + K} - W(s)\frac{1}{s + 5 + K}$$

Steady-State Error – Example 4

- Error due to a constant disturbance can be found by applying the ***final value theorem***

$$e_{ss,w} = \lim_{s\to 0} s\left(-W(s)\frac{1}{s+5+K}\right)$$

$$e_{ss,w} = \lim_{s\to 0}\left(-s\frac{1}{s}\frac{1}{s+5+K}\right) = -\frac{1}{5+K}$$

- We can calculate the ***required gain*** for 1% error

$$\left|e_{ss,w}\right| = 0.01 = \frac{1}{5+K} \quad\longrightarrow\quad \boxed{K = 95}$$

- At this gain value, the ***error due to a constant reference*** is

$$e_{ss,r} = \lim_{s\to 0}\left(s\frac{1}{s}\frac{s+5}{s+5+K}\right) = \frac{5}{100} \quad\longrightarrow\quad \boxed{5\%}$$