Transfer Function → State Space (order of numerator=order of denominator)

- Controllable Canonical Form
- Observable Canonical Form

If the order of the numerator is equal to the order of the denominator, it becomes more difficult to convert from a system transfer function to a state space model. This document shows how to do this for a 3rd order system. The technique easily generalizes to higher order.

Controllable Canonical Form (CCF)

Consider the third order differential transfer function:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

We start by multiplying by Z(s)/Z(s) and then solving for Y(s) and U(s) in terms of Z(s). We also convert back to a differential equation.

$$\begin{aligned} Y(s) &= \left(b_0 s^3 + b_1 s^2 + b_2 s + b_3\right) Z(s) & \qquad y &= b_0 \dddot{z} + b_1 \ddot{z} + b_2 \dot{z} + b_3 z \\ U(s) &= \left(s^3 + a_1 s^2 + a_2 s + a_3\right) Z(s) & \qquad u &= \dddot{z} + a_1 \ddot{z} + a_2 \dot{z} + a_3 z \end{aligned}$$

We can now choose z and its first two derivatives as our state variables

$$\begin{aligned} q_1 &= z & \qquad \dot{q}_1 &= \dot{z} = q_2 \\ q_2 &= \dot{z} & \qquad \dot{q}_2 &= \ddot{z} = q_3 \\ q_3 &= \ddot{z} & \qquad \dot{q}_3 &= \dddot{z} = u - a_1 \ddot{z} - a_2 \dot{z} - a_3 z \\ & & &= u - a_1 q_3 - a_2 q_2 - a_3 q_1 \end{aligned}$$

Now we just need to form the output

$$y = b_0 \dddot{z} + b_1 \ddot{z} + b_2 \dot{z} + b_3 z$$

Unfortunately, the third derivative of z is not a state variable or an input, so this is not a valid output equation. However, we can represent the term as a sum of state variables and outputs:

$$\dddot{z} = u - a_1\ddot{z} - a_2\dot{z} - a_3 z$$

and

$$\begin{aligned} y &= b_0\left(u - a_1\ddot{z} - a_2\dot{z} - a_3 z\right) + b_1\ddot{z} + b_2\dot{z} + b_3 z \\ &= b_0 u + \left(b_1 - a_1 b_0\right)\ddot{z} + \left(b_2 - a_2 b_0\right)\dot{z} + \left(b_3 - a_3 b_0\right)z \end{aligned}$$

From these results we can easily form the state space model:

$$\dot{\mathbf{q}} = \mathbf{Aq} + \mathbf{B}u; \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}; \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$y = \mathbf{Cq} + Du \quad \mathbf{C} = \left[\left(b_3 - a_3 b_0\right) \ \left(b_2 - a_2 b_0\right) \ \left(b_1 - a_1 b_0\right)\right] \quad D = b_0$$

In this case, the order of the numerator of the transfer function was less than that of the denominator. If they are equal, the process is somewhat more complex. A result that works in all cases is given below; the details are here.

Observable Canonical Form (OCF)

Consider the third order differential transfer function:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

We can convert this to a differential equation and solve for the highest order derivative of y:

$$\left(s^3 + a_1 s^2 + a_2 s + a_3\right)Y(s) = \left(b_0 s^3 + b_1 s^2 + b_2 s + b_3\right)U(s)$$

$$s^3 Y(s) = \left(b_0 s^3 + b_1 s^2 + b_2 s + b_3\right)U(s) - \left(a_1 s^2 + a_2 s + a_3\right)Y(s)$$

$$\dddot{y} = b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u - a_1\ddot{y} - a_2\dot{y} - a_3 y$$

Now we integrate twice (the reason for this will be apparent soon), and collect terms according to order of the integral (this includes bringing the first derivative of u to the left hand side):

$$\dot{y} = b_0\dot{u} + b_1u + b_2\int u\cdot dt + b_3\int\int u\cdot dt\cdot dt - a_1y - a_2\int y\cdot dt - a_3\int\int y\cdot dt\cdot dt$$
$$\dot{y} - b_0\dot{u} = b_1u - a_1y + \int(b_2u - a_2y)dt + \int\int(b_3u - a_3y)\cdot dt\cdot dt$$

Without an justification we choose y-b_0u as our first state variable

$$q_1 = y - b_0u \qquad \dot{q}_1 = \dot{y} - b_0\dot{u} = -a_1y + \int(b_2u - a_2y)dt + \int\int(b_3u - a_3y)\cdot dt\cdot dt$$

Looking at the right hand side of the differential equation we note that y=q_1 and we call the two integral terms q_2:

$$q_2 = \int(b_2u - a_2y)dt + b_2\int\int(b_3u - a_3y)\cdot dt\cdot dt$$
$$\dot{q}_1 = \dot{y} - b_0\dot{u} = b_1u - a_1y + q_2$$

This isn't a valid state equation because it has "y" on the right side (recall that only state variables and inputs are allowed). We can get rid of it by noting:

$$y = q_1 + b_0u$$

so

$$\begin{aligned}\dot{q}_1 &= b_1u - a_1(q_1 + b_0u) + q_2\\ &= -a_1q_1 + q_2 + (b_1 - a_1b_0)u\end{aligned}$$

This is our first state variable equation.

Now let's examine q_2 and its derivative:

$$q_2 = \int(b_2u - a_2y)dt + b_2\int\int(b_3u - a_3y)\cdot dt\cdot dt$$
$$\dot{q}_2 = b_2u - a_2y + \int(b_3u - a_3y)\cdot dt$$

Again we note that y=q_1+b_0u and we call the integral terms q_3:

$$q_3 = \int(b_3u - a_3y)\cdot dt$$
$$\dot{q}_2 = b_1u - a_2q_1 + q_3$$

This is our second state variable equation.

Now let's examine q_3 and its derivative:

$$\begin{aligned}\dot{q}_3 &= b_3u - a_3y \\ &= b_3u - a_3\left(q_1 + b_0u\right) \\ &= -a_3q_1 + \left(b_3 - a_3b_0\right)\end{aligned}$$

This is our third, and last, state variable equation.

Our state space model now becomes:

$$\dot{\mathbf{q}} = \mathbf{Aq} + \mathbf{B}u = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix}\mathbf{q} + \begin{bmatrix} b_1 - a_1b_0 \\ b_2 - a_2b_0 \\ b_3 - a_3b_0 \end{bmatrix}u$$

$$y = \mathbf{Cq} + Du = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{q} + b_0 \cdot u$$

Here is a good reference that does the same derivations from a different perspective: http://www.ece.rutgers.edu/~gajic/psfiles/canonicalforms.pdf

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References