Steady-State Error For Non-Unity Feedback Systems $(H(s) \neq 1)$

A. Overview

For unity feedback systems with $H(s) = 1$, the output of the summing junction in the normal single-loop feedback configuration is the difference between the reference input $R(s)$ and the output $C(s)$. Assuming that the reference input is the desired value of the output, then the output of the summing junction is the system error $E(s)$. The expression for the steady-state error, $e_{ss} = \lim_{t\to\infty} [e(t)]$ is easily derived for the unity feedback case; this development is shown on the web page http://ece.gmu.edu/~gbeale/ece_421/ess_01.html.

When the system does not have unity feedback, that is, $H(s) \neq 1$, then the output of the summing junction is not the difference between the reference input and the actual output. However, we will still define the error as $E(s) = R(s) - C(s)$. Unfortunately, that signal does not appear in the non-unity feedback block diagram, shown in the top diagram of Fig. 1. The signal $E(s)$ is shown in the second diagram in the figure.

Two methods will be used to derive the expression for steady-state error for the non-unity feedback case. The first method will proceed directly from the second block diagram in the figure. The second method will use block diagram manipulation from the top diagram in the figure to convert the system into an equivalent unity feedback system. We will make the same assumptions on the types of input signals that will be considered that were made for unity feedback systems, namely that $R(s) = A/s^q$.

B. Method 1

The system is described by the top diagram in Fig. 1. The forward transfer function $G(s)$ is given by

$$G(s) = \frac{KN_g(s)}{D_g(s)} = \frac{K_x N_{g1}(s)}{s^N D_{g1}(s)}, \qquad \begin{matrix} N_{g1}(0) = 1 \\ D_{g1}(0) = 1 \end{matrix} \tag{1}$$

where N is the number of open-loop poles in $G(s)$ that occur at $s = 0$. The feedback transfer function $H(s)$ is given by

$$H(s) = \frac{K_{FB}N_h(s)}{D_h(s)} = \frac{K_h N_{h1}(s)}{D_{h1}(s)}, \qquad \begin{matrix} N_{h1}(0) = 1 \\ D_{h1}(0) = 1 \end{matrix} \tag{2}$$

The feedback block $H(s)$ is assumed to have no poles or zeros at the origin. The final forms shown for both $G(s)$ and $H(s)$ are the time constant form, so the only terms that will impact the steady-state error will be the gains K_x and K_h and the N poles of $G(s)$ at the origin.

As shown in the second diagram in Fig. 1, the error signal is given by $E(s) = R(s) - C(s)$, which becomes

$$E(s) = R(s) - T_{CL}(s)R(s) = \left[1 - \frac{G(s)}{1 + G(s)H(s)}\right] R(s) = \left[\frac{1 + G(s)H(s) - G(s)}{1 + G(s)H(s)}\right] R(s) \tag{3}$$

Assuming that the closed-loop system is stable, the steady-state error can be obtained by applying the Final Value Theorem to $E(s)$. This is

$$e_{ss} = \lim_{t\to\infty} [e(t)] = \lim_{s\to 0} [sE(s)] = \lim_{s\to 0} \left\{ s \left[\frac{1 + G(s)H(s) - G(s)}{1 + G(s)H(s)}\right] R(s) \right\} \tag{4}$$

With the reference input restricted to the class of signals represented by $R(s) = A/s^q$, the expression for e_{ss} can be written as

$$e_{ss} = A \cdot \lim_{s\to 0} \left\{ s^{(1-q)} \left[\frac{1 + G(s)H(s) - G(s)}{1 + G(s)H(s)}\right] \right\} \tag{5}$$

If we substitute the time constant forms for $G(s)$ and $H(s)$ from (1) and (2) into (5), we get the following results.

$$e_{ss} = A \cdot \lim_{s \to 0} \left\{ s^{(1-q)} \left[\frac{1 + \frac{K_x N_{g1}(s)}{s^N D_{g1}(s)} \cdot \frac{K_h N_{h1}(s)}{D_{h1}(s)} - \frac{K_x N_{g1}(s)}{s^N D_{g1}(s)}}{1 + \frac{K_x N_{g1}(s)}{s^N D_{g1}(s)} \frac{K_h N_{h1}(s)}{D_{h1}(s)}} \right] \right\} = A \cdot \lim_{s \to 0} \left\{ s^{(1-q)} \left[\frac{1 + \frac{K_x K_h}{s^N} - \frac{K_x}{s^N}}{1 + \frac{K_x K_h}{s^N}} \right] \right\} \tag{6}$$

where the last expression makes use of the fact that $N_{g1}(0) = D_{g1}(0) = N_{h1}(0) = D_{h1}(0) = 1$. Multiplying numerator and denominator of (6) by s^N, we get

$$e_{ss} = A \cdot \lim_{s \to 0} \left\{ s^{(1-q)} \left[\frac{s^N + K_x K_h - K_x}{s^N + K_x K_h} \right] \right\} = A \cdot \lim_{s \to 0} \left[\frac{s^{(N+1-q)}}{s^N + K_x K_h} + \frac{K_x (K_h - 1) s^{(1-q)}}{s^N + K_x K_h} \right] \tag{7}$$

Equation (7) is the general expression for steady-state error with non-unity feedback.

Clearly, if $K_h = \lim_{s \to 0} [H(s)] = 1$, the expression for e_{ss} in (7) is identical to the expression for the unity feedback case. Thus, the non-unity feedback configuration is equivalent—from the standpoint of steady-state error—to the unity feedback configuration if $\lim_{s \to 0} [H(s)] = 1$.

If the reference input is a step input, $q = 1$. If $N \geq 1$, the steady-state error becomes

$$e_{ss} = A \cdot \lim_{s \to 0} \left[\frac{s^N + K_x K_h - K_x}{s^N + K_x K_h} \right] = A \left(\frac{K_h - 1}{K_h} \right) \tag{8}$$

so $e_{ss} = 0$ when $K_h = 1$, as expected. If $K_h < 1$, the steady-state error is negative, which means that the output is larger than the input. If $K_h = 0$, the steady-state error becomes infinity; the feedback loop is broken in steady-state, $G(s)$ has N integrators, so the output is unbounded for the step input.

If $q > 1$, the steady-state error is

$$e_{ss} = A \cdot \lim_{s \to 0} \left[\frac{s^{(N+1-q)}}{s^N + K_x K_h} + \frac{1}{s^{(q-1)}} \cdot \frac{K_x (K_h - 1)}{s^N + K_x K_h} \right] \tag{9}$$

so $e_{ss} = \infty$ for $q > 1$ unless $K_h = 1$, regardless of the value of N. Thus, non-unity feedback systems will always have infinite steady-state error if the reference input is any signal of higher order than a step function if $K_h \neq 1$.

C. Method 2

A simple identity can be used to convert a non-unity feedback system into an equivalent unity feedback system. This identity is $H(s) = H(s) - 1 + 1$, so the first diagram in Fig. 1 can be converted into the third diagram. The inner loop has a feedback block of $H(s) - 1$, and the outer loop has unity feedback. The error signal is the output of the left-most summing junction in the third diagram. The inner loop can be reduced to a single block with transfer function

$$G_1(s) = \frac{G(s)}{1 + G(s) [H(s) - 1]} \tag{10}$$

The equivalent system, shown in the bottom diagram of Fig. 1 has the standard unity feedback configuration, with $G_1(s)$ being the forward transfer function. Once this conversion has been done, the normal unity feedback equations for steady-state error can be applied directly to $G_1(s)$, that is,

$$e_{ss} = A \cdot \lim_{s \to 0} \left[\frac{s^{(N_1+1-q)}}{s^{N_1} + K_{x1}} \right] \tag{11}$$

where N_1 is the number of poles of $G_1(s)$ at the origin and
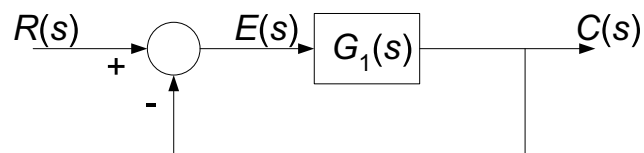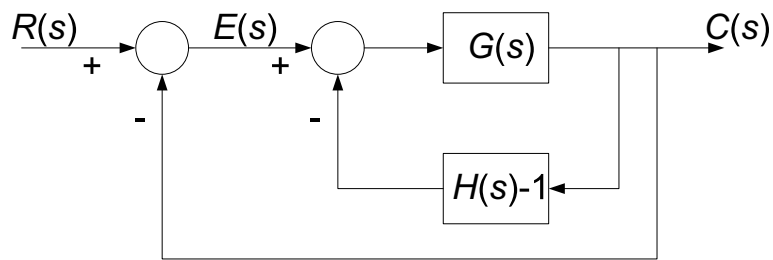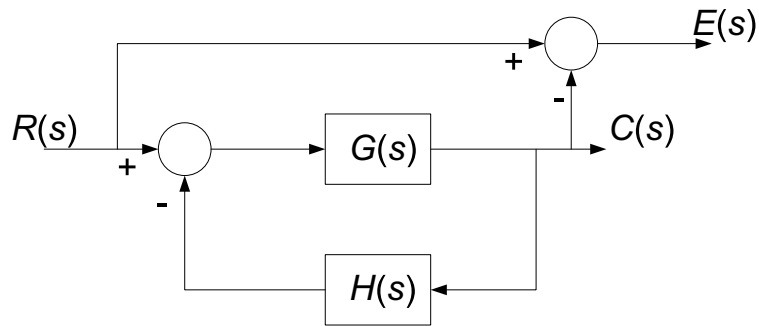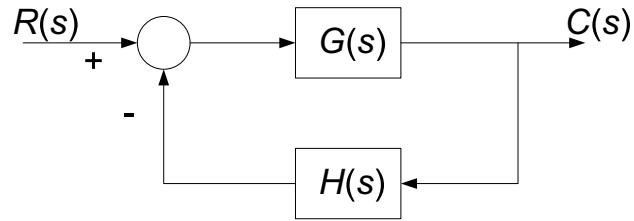
$$K_{x1} = \lim_{s \to 0} [G_1(s)] \tag{12}$$

Fig. 1.   Block diagrams for non-unity feedback steady-state error.